Finding Departure-Time Nash Equilibrium in a Generic Bottleneck Model: An Heuristic Algorithm

Master's Thesis

Student: Lucas JAVAUDIN, Supervisor: André DE PALMA, Jury: Robert GARY-BOBO

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Contents

Introduction

Context

- The bottleneck model, introduced by Vickrey (1969), is the most popular model to study rush-hour departure-time choice
- The original model considers a single-road network with a continuum of identical commuters with linear preferences
- Many extensions have been proposed with nonlinear preferences, heterogeneous commuters, multiple-road networks, etc.
- There is no analytical method able to solve the model in a general case

Introduction

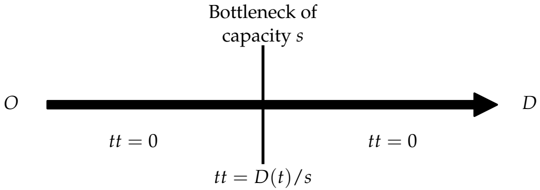
Contributions

- I propose an heuristic algorithm to find the equilibrium of the bottleneck model
- I show that the algorithm replicate very well the solutions found with analytical methods
- The algorithm can find the equilibrium in a model with heterogeneity, multiple-road network and endogeneity
- I identify three novel properties on models which are too complex to be solved with analytical methods

Bottleneck Model

Supply-Side

- Single-road network with a bottleneck of capacity s



- Total travel time is

$$T(t) = \frac{D(t)}{s}$$

- Queue length is

$$D(t) = \max\left(0, \sup_{\tau \in \{t_0, t\}} \int_{\tau}^{t} \left(r(u) - s\right) du\right)$$

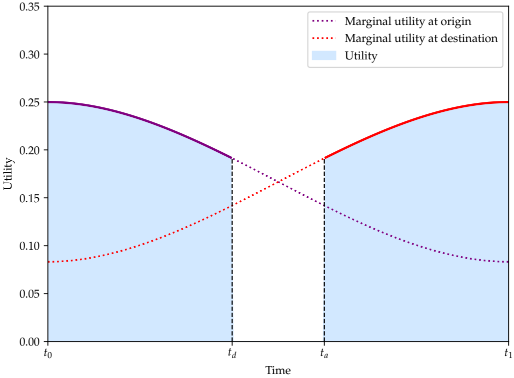
where $r(t)$ is departure rate at time t

Bottleneck Model

Demand-Side

- Total utility given departure time t_d and arrival time t_a is

$$U(t_d, t_a) = \int_{t_0}^{t_d} u^o(t)dt + \int_{t_a}^{t_1} u^d(t)dt$$

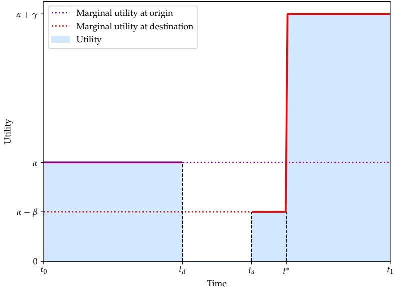


Bottleneck Model

α-β-γ Model

- t^*: desired arrival time
- α: value of time
- β: penalty for early arrival
- γ: penalty for late arrival

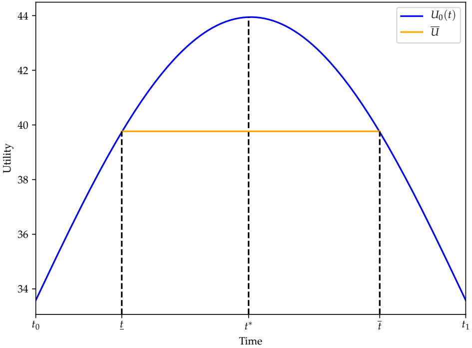
$$U(t_d, t_a) = -[\alpha(t_a - t_d) + \beta(t^* - t_a)_+ + \gamma(t_a - t^*)_+]$$



Bottleneck Model

Equilibrium

- Free-flow utility is $U_0(t) = U(t,t) = \int_{t_0}^{t} u^o(u)du + \int_{t}^{t_1} u^d(u)du$
- At equilibrium, all N commuters are leaving origin between $\underline{t}$ and $\overline{t}$ and have the same utility level $\overline{U}$
- Utility at $\underline{t}$ and $\overline{t}$ is equal to free-flow utility



Bottleneck Model

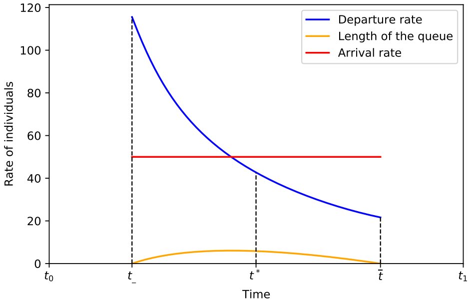
Equilibrium

- Equilibrium travel times are implicitly given by

$$U_0(t) - \overline{U} = \int_{t}^{t+T(t)} u^d(s)ds, \quad \forall t \in [\underline{t}, \overline{t}]$$

- Departure rate of individuals are derived from queue length equation:

$$r(t) = s \cdot u^o(t)/u^d(t + T(t)), \quad t \in [\underline{t}, \overline{t}]$$



Bottleneck Model

Discretization

- The model presented above assume a continuous strategy space (i.e. continuous time) and a continuum of commuters
- Many papers with numeric methods assume a discrete strategy set (e.g. papers on day-to-day dynamics)
- Otsubo and Rapoport (2008) assume indivisible commuters
- I assume a discrete strategy set and discrete commuters
- The results of the continuous and discrete model are very close for a large number of commuters and periods

Details

Day-to-Day Dynamics

Introduction

- Day-to-day dynamics models study the convergence of the bottleneck model to an equilibrium
- These models try to replicate the behaviors of commuters from day to day
- Iryo (2008) proves the instability of the equilibrium with continuous iterations
- Guo et al. (2018) proves the instability of the equilibrium with discrete iterations
- Both papers use the proportional swap mechanism
- These models are limited to homogeneous commuters in a single-road network

Day-to-Day Dynamics

Proportional Swap Mechanism (Smith, 1984)

- Departure-time space is discretized and commuter space is continuous
- The population shift from departure $t_i \in \mathcal{T}$ to departure $t_j \in \mathcal{T}$ from one iteration to the next is

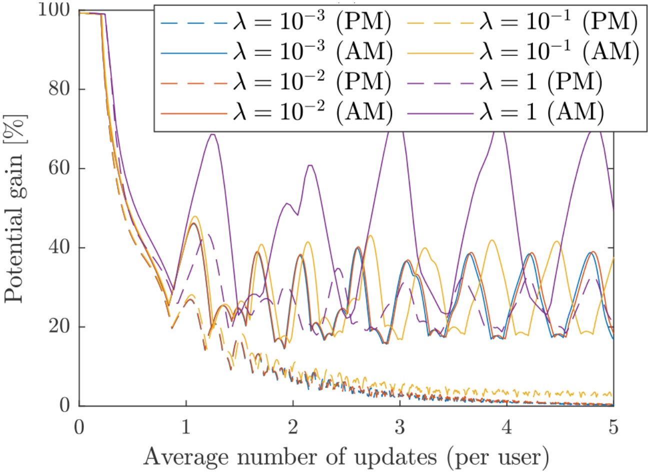
$$\lambda r(t_i)\left[U_j(r) - U_i(r)\right]_+$$

where

- λ defines the step size,
- $r(t_i)$ is the departure rate at time t_i
- $U_i(r)$ is the utility associated with departure t_i

Day-to-Day Dynamics

Convergence results (Lamotte and Geroliminis, 2020)



Algorithm

Potential

- Given departure times $\mathbf{t}^* = \{t_1^*, \ldots, t_N^*\}$, the potential φ_i of a commuter i is the relative difference between the maximum utility and the current utility of the commuter

$$\varphi_i(\mathbf{t}^*) = \frac{\max_t U_i(t) - U_i(t_i^*)}{U_i(t_i^*)}$$

- The average potential $\frac{1}{N}\sum_i \varphi_i(\mathbf{t}^*)$ can be used to measure distance of a state $\mathbf{t}^*$ to equilibrium

Algorithm

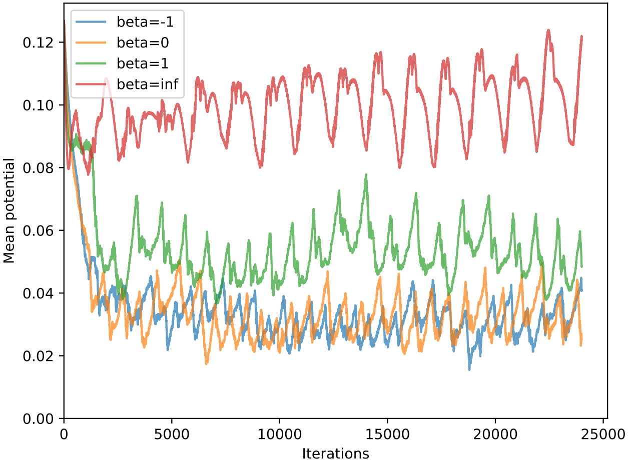
Naive Algorithm

1. Initialize random departure times $\mathbf{t}^0$ and set iteration counter $\tau = 0$.
2. For each individual i, compute the utility $U_i^\tau(t)$ that she can get at any departure time t, given the departure times t_{-i}^τ of the other individuals.
3. Compute the potential φ_i^τ of each individual i.
4. Randomly select an individual with probabilities proportional to $(\varphi_i^\tau)^\beta$.
5. Switch the selected individual to her best departure time: $t_i^{\tau+1} = \arg\max_t \; U_i^\tau(t)$.
6. Stop the algorithm if some convergence criterion is met; otherwise, set $\tau = \tau + 1$ and go back to 2.

Example

Algorithm

Naive Algorithm



Algorithm

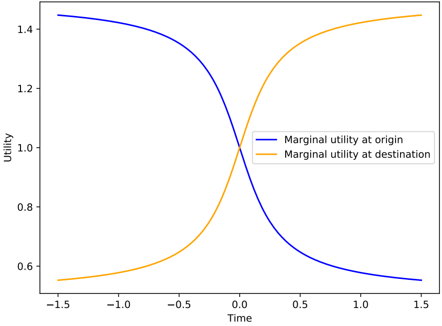
Main Algorithm

1. Initialize random departure times $\mathbf{t}^0$ and set iteration counter $\tau = 0$.
2. For each individual i, compute the utility $U_i^\tau(t)$ that she can get at any departure time t, given the departure times t_{-i}^τ of the other individuals.
3. Compute the potential φ_i^τ of each individual i.
4. Randomly select an individual with probabilities proportional to $(\Phi_i^\tau)^\beta$.
5. *Sort the departure times of the selected individual by order of decreasing utility and randomly select one departure time $\hat{t}$ in the first quantile of order q. Switch the selected individuals to this departure time, i.e. $t_i^\tau = \hat{t}$.*
6. *Compute some criterion measuring distance to equilibrium. If the switch does not improve this criterion, then revert the switch by putting the switched individual back to her previous departure time.*
7. Stop the algorithm if some convergence criterion is met; otherwise, set $\tau = \tau + 1$ and go back to 2.

Simulations

Setup

- $N = 1200,\ m + 1 = 181,\ t_0 = -1.5,\ t_1 = 1.5$
- Marginal utility at origin is $u^o(t) = 1 - \frac{\tan^{-1}(4t)}{\pi}$ and marginal utility at destination is $u^d(t) = 1 + \frac{\tan^{-1}(4t)}{\pi}$

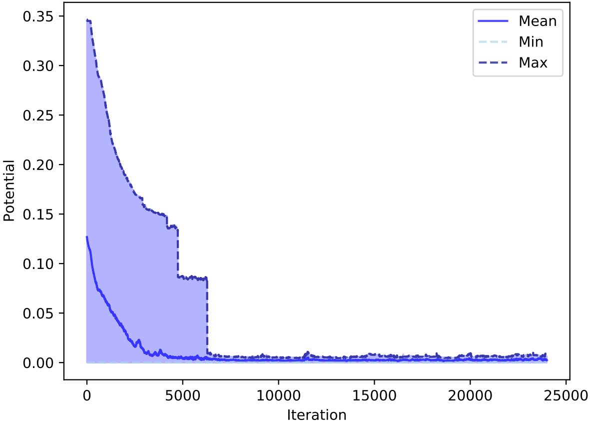


Simulations

Potential Convergence

Calibration

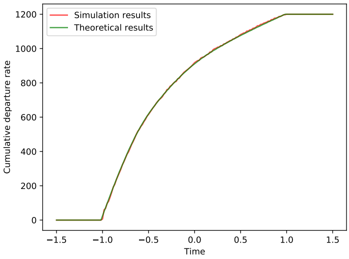
$\beta = 0, \; q = 20\%$



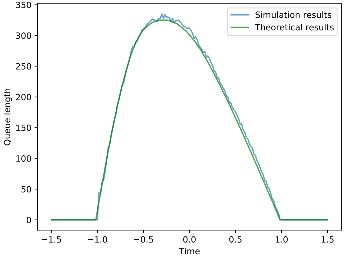
Simulations

Results

Departure rate



Queue length



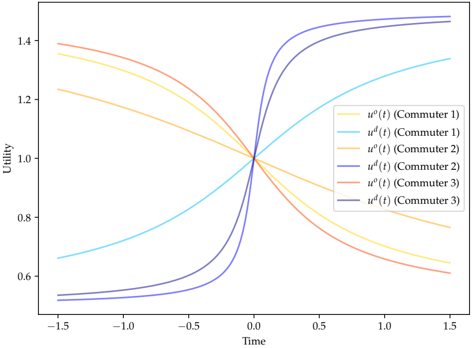
Other preferences

Heterogeneity

Setup

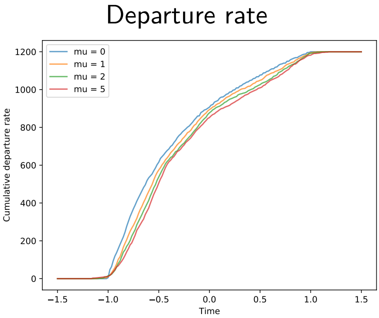
Marginal utility at origin is $u_i^o(t) = 1 - \frac{\tan^{-1}(4 \cdot v_i^o \cdot t)}{\pi}$ and marginal utility at destination is $u_i^d(t) = 1 + \frac{\tan^{-1}(4 \cdot v_i^d \cdot t)}{\pi}$ where

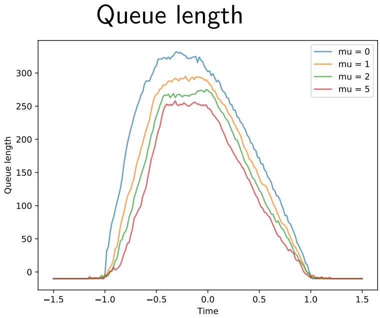
$$\left(\ln(v_i^o), \ln(v_i^d)\right)^T \sim \mathcal{N}\left((0,0)^T, \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\right)$$



Heterogeneity

Results





With higher commuter heterogeneity, rush hour is shorter and congestion is smaller

Endogeneity

Setup (from Fosgerau and Small, 2017)

- Marginal utility at origin is $u^o(t) = [x^o(t)]^{\pi_o}$ and marginal utility at destination is $u^d(t) = [x^d(t)]^{\pi_d}$, with

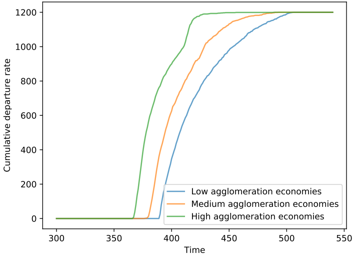
$$U(t_d, t_a) = 2\ln\left(\int_{t_0}^{t_d} u^o(t)dt\right) + \ln\left(\int_{t_a}^{t_1} u^d(t)dt\right)$$

- π_o and π_d represent the intensity of the agglomeration economies at origin and at destination

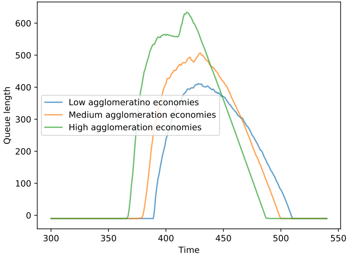
Endogeneity

Results

Departure rate



Queue length

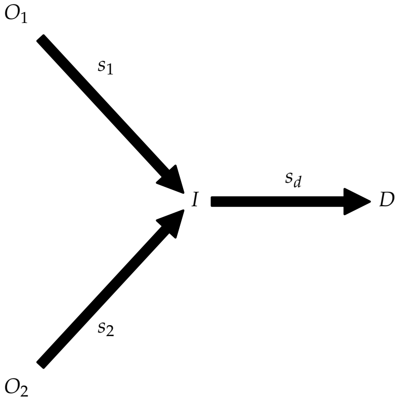


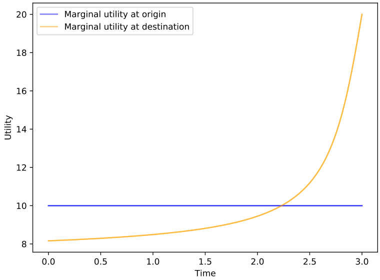
With higher agglomeration economies, rush hour is earlier and congestion is larger

Road Network

Setup (from Arnott et al., 1993)

- Network with two upstream bottlenecks (capacity s_1 and s_2) and one downstream bottleneck (capacity s_d)
- $n_1 = 150$ commuters leaving from origin O_1 and $n_2 = 300$ commuters leaving from origin O_2





Road Network

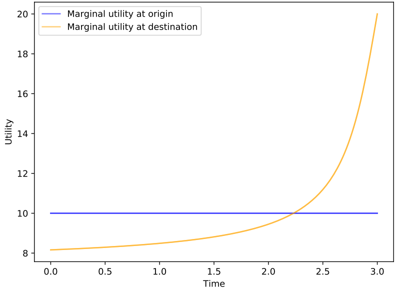
Paradox

- With α-β-γ preferences, Arnott et al. show that the derivative of total cost is positive with s_2 under the following condition

$$(1+\nu)(1-\theta) < \frac{s_d}{s_2} < \max\left(1, 1-\theta+\sqrt{\nu(\nu+\theta)(1-\theta)}\right)$$

where $\theta = \beta/\alpha$ and $\nu = n_1/n_2$

- We test if the paradox holds with nonlinear preferences



Endogeneity

Results

s_2	Average potential	Total cost
2	0.98%	40015
3	0.32%	31927
4	0.38%	32946
5	0.33%	32191

The paradox still holds with nonlinear preferences for some values of s_2

Conclusion

- I proposed an algorithm able to find the equilibrium in a general bottleneck model
- The algorithm can be used to identify novel properties in intractable models
- Directions for future research:
  - Calibration
  - Different forms of heterogeneity
  - Policies
  - Joint morning-evening commute choice

Discrete Bottleneck Model

Setup

- There are N commuters and $m+1$ time periods
- The strategy set is

$$\mathcal{T} = \{t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots, t_0 + (m-1)\Delta t, t_1\}$$

where $\Delta t = (t_1 - t_0)/m$

- Queue length is defined by $D(t_0) = \max\big(r(t) - s, 0\big)$ and

$$D(t) = \max\big(D(t-1) + r(t) - s, 0\big), \quad \forall t \in \mathcal{T},\ t > t_0$$

where $r(t)$ is the number of commuters leaving from origin at time t

Discrete Bottleneck Model

Travel Time Probability

- If $D(t_d - 1) = 0$ and $r(t_d) \leq s$, $T(t_d, t_d) = 1$ and $T(t_d, t) = 0$, for any $t \neq t_d$
- If $D(t_d - 1) = 0$ and $r(t_d) > s$,

$$T(t_d, t_a) = \begin{cases} 0 & \text{if } t_a < t_d \\ \frac{s}{r(t_d)} & \text{if } t_a \in [t_d, \tilde{t}) \\ \frac{r(t_d) \mod s}{r(t_d)} & \text{if } t_a = \tilde{t} \\ 0 & \text{if } t_a > \tilde{t} \end{cases},$$

where $\tilde{t} = t_d + \lfloor r(t)/s \rfloor$ is the time at which the last commuter is served

Discrete Bottleneck Model

Travel Time Probability

- If $D(t_d - 1) > 0$,

$$T(t_d, t_a) = \begin{cases} 0 & \text{if } t_a < \hat{t} \\ \frac{s - R(t_d)}{r(t_d)} & \text{if } t_a = \hat{t} \\ \frac{s}{r(t_d)} & \text{if } t_a \in (\hat{t}, \tilde{t}) \\ \frac{\left(r(t_d) - s + R(t_d)\right) \mod s}{r(t_d)} & \text{if } t_a = \tilde{t} \\ 0 & \text{if } t_a > \tilde{t} \end{cases},$$

where

- $\hat{t} = t_d + \lfloor D(t_d - 1)/s \rfloor$ the time at which the last commuter in the queue at time $t_d - 1$ is served;
- $R(t_d) = D(t_d - 1) \mod s$ the number of commuters in the queue at time $t_d - 1$ who are served at time $\hat{t}$;
- $\tilde{t} = \hat{t} + \lfloor (r(t_d) - s + R(t_d))/s \rfloor$ the time at which the last commuter who arrived at the bottleneck at time t_d is served.

Discrete Bottleneck Model

Example of Travel Time Probability

$s = 5,\ D(t_d - 1) = 7,\ r(t_d) = 15$

t	Capacity s	# served who left before t_d	# served who left at t_d	$T(t_d, t)$
t_d	5	5	0	0
$t_d + 1$	5	2	3	$3/15$
$t_d + 2$	5	0	5	$5/15$
$t_d + 3$	5	0	5	$5/15$
$t_d + 4$	5	0	2	$2/15$

Discrete Bottleneck Model

Utility and Nash Equilibrium

- Utility when leaving at time t is

$$U_i(t) = \sum_{\tau \leq t} u_i^o(\tau) + \sum_{\tau \geq t} u_i^d(\tau) F(t, \tau),$$

where

$$F(t, \tau) = \sum_{u \leq \tau} T(t, u)$$

- An equilibrium of this model is a set of departure-time values $\mathbf{t}^* = \{t_1^*, \ldots, t_N^*\} \in \mathcal{T}^N$ such that no commuter i can increase her utility by switching from departure time t_i^* to $t \neq t_i^*$, while departure times t_{-i}^* of the other commuters are fixed

Go back

Algorithm Example

Possible Equilibrium

$s = 2$

t	$u^o(t)$	$u^d(t)$	$U_{\text{freeflow}}(t)$
0	3	0	11
1	3	3	14
2	4	4	15
3	2	1	13
4	0	0	12

t	$r(t)$	$U_t(t)$	$\max_\tau U_t(\tau)$	$\varphi(t)$
0	0	(11)	13	(2/11)
1	2	14	14	0
2	4	13	13	0
3	0	(11)	13	(2/11)
4	0	(12)	13	(1/12)

Algorithm Example

Iteration 0

t	$u^o(t)$	$u^d(t)$	$U_{\text{freeflow}}(t)$
0	3	0	11
1	3	3	14
2	4	4	15
3	2	1	13
4	0	0	12

t	$r(t)$	$U_t(t)$	$\arg\max_\tau U_t(\tau)$	$\max_\tau U_t(\tau)$	$\varphi(t)$
0	1	11	1	14	**3/11**
1	1	14	1	14	0
2	2	15	2	15	0
3	1	13	1	14	1/13
4	1	12	1	14	1/6

Algorithm Example

Iteration 1

t	$u^o(t)$	$u^d(t)$	$U_{\text{freeflow}}(t)$
0	3	0	11
1	3	3	14
2	4	4	15
3	2	1	13
4	0	0	12

t	$r(t)$	$U_t(t)$	$\arg\max_\tau U_t(\tau)$	$\max_\tau U_t(\tau)$	$\varphi(t)$
0	0	(11)	·	·	·
1	2	14	1	14	0
2	2	15	2	15	0
3	1	13	2	13.67	2/39
4	1	12	2	13.67	**5/36**

Algorithm Example

Iteration 2

t	$u^o(t)$	$u^d(t)$	$U_{\text{freeflow}}(t)$
0	3	0	11
1	3	3	14
2	4	4	15
3	2	1	13
4	0	0	12

t	$r(t)$	$U_t(t)$	$\arg\max_\tau U_t(\tau)$	$\max_\tau U_t(\tau)$	$\varphi(t)$
0	0	(11)	·	·	·
1	2	14	1	14	0
2	3	13.67	2	13.67	0
3	1	13	1, 2 or 3	13	0
4	0	(12)	·	·	·

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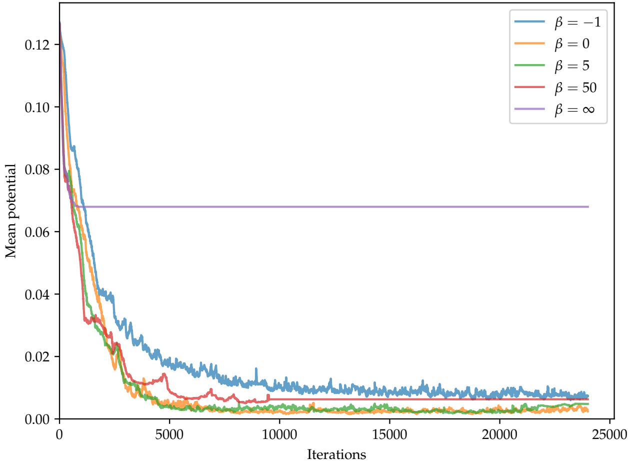
Simulations

Convergence of Potential with β

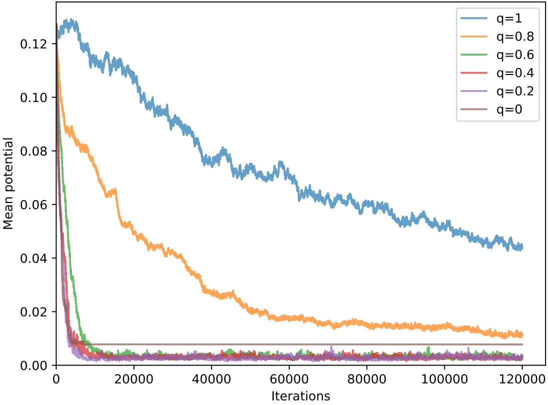
β defines how likely it is to switch individuals with high potential



Simulations

Convergence of Potential with q

q defines how the new departure time of the switched individual is selected



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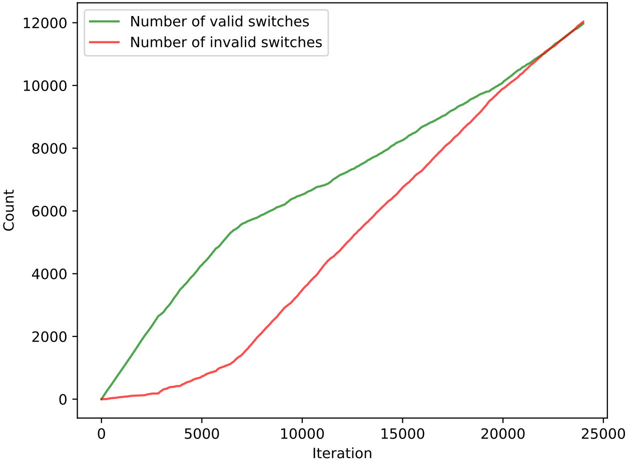
Simulations
Switch validity

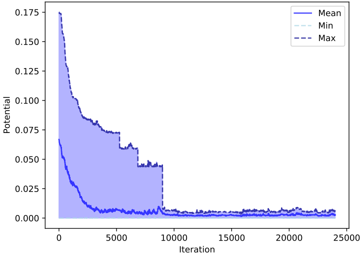
A switch is valid if it improves some distance to equilibrium



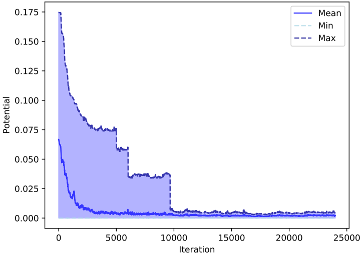
Simulations

Potential Convergence with Morning and Evening Commute

Morning commute

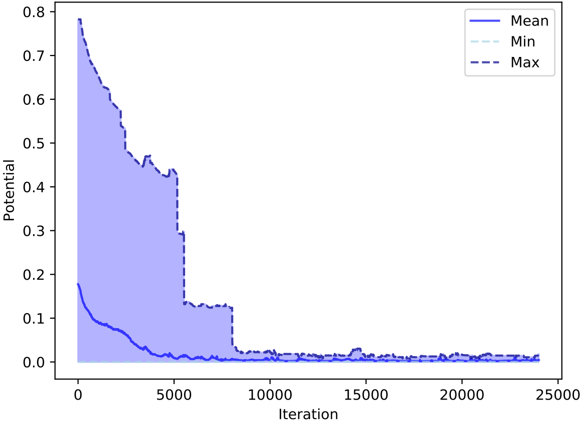


Evening commute



Simulations

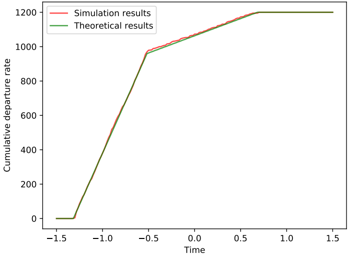
Potential Convergence with α-β-γ Preferences



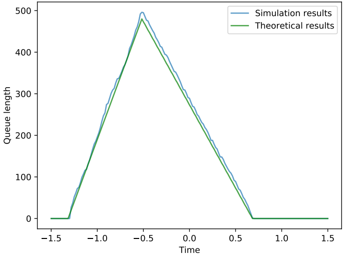
Simulations

Results with α-β-γ Preferences

Departure rate



Queue length



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